Suggested solution of HW5

Ch1-Q7: (a) If $|w| < 1$, consider the holomorphic map $z \mapsto (w-z)/(1-\bar{w}z)$ on $\mathbb{D}$. By maximum principle,

$$\left|\frac{w-z}{1-\bar{w}z}\right| \leq \sup_{\mathbb{D}} \left|\frac{w-z}{1-\bar{w}z}\right|.$$

On $\partial\mathbb{D}$, $\bar{z} = 1/z$,

$$\frac{w-z}{1-\bar{w}z}\frac{\bar{w}-\bar{z}}{1-w\bar{z}} = \frac{w-z}{1-\bar{w}z}\frac{z\bar{w}-1}{z-w} = 1.$$

Conclusion followed. (I am sorry that I am too lazy to compute it directly.)

(b) i. Clearly, F is holomorphic. If $|F(a)| = 1 = \sup_{\mathbb{D}} |F|$ for some $a \in \mathbb{D}$, by maximum principle or open mapping mapping, F is constant map which is clearly impossible. So, $F : \mathbb{D} \to \mathbb{D}$.

ii. By direct computation.

iii. As illustrate in part (a).

iv. It can be checked that the inverse of F is given by

$$F^{-1}(z) = \frac{w+z}{1+\bar{w}z} : \mathbb{D} \to \mathbb{D}.$$

Ch2-Q7: Consider $g(z) = \frac{f(z)-f(-z)}{d}$ on $\mathbb{D}$ in which $g(0) = 0$ and $|g(z)| \leq 1$. By Schwarz Lemma,

$$1 \geq |g'(0)| = \frac{2}{d}\left|f'(0)\right|.$$

Ch8-Q1: Suppose $\alpha = f'(a) \neq 0$. Writing $f(z) = h(z)(z-a) + f(a)$. There exists a ball $B(a, r)$ such that on $B(a, r)$,

$$|h(z) - \alpha| < \frac{|\alpha|}{4}.$$

Thus, on $\partial B(a, r)$,

$$\begin{aligned}
|f(z) - \alpha(z-a)| &= |h(z) - \alpha||z-a| \\
&< r \cdot \frac{|\alpha|}{4} \\
&< r \cdot \frac{|\alpha|}{2} < |h(z)||z-a|.
\end{aligned}$$

For all $w \in B(a, r)$, $f(z) - w$ has same number of zeros with $\alpha(z-a) - w$ in $B(a, r)$. So it is one-one as $\alpha(z-a) - w$ is linear.

Suppose f is local bijection. If $f'(a) = 0$ for some $a \in U$, without loss generality, we assume $f(a) = 0, a = 0$. So we can write $f(z) = z^m h(z)$ where $h(z)$ is holomorphic, $h(0) = \alpha \neq 0$ and $m \geq 2$. There exists a ball $B(0, r)$ such that

$$|h(z) - \alpha| < |\alpha|/4, \quad \text{on } B(a, r).$$

So, on $\partial B(a, r)$

$$|f(z) - \alpha z^m| = |h(z) - \alpha| \cdot r^m < |\alpha| r^m/4 < |\alpha||z|^m.$$

By Rocuh's theorem, $f(z)$ has same number of zeros with αz^m, which is clearly not injective when $m \geq 2$.

Ch8-Q4: Consider the biholomorphic map from $\mathbb{D}$ to $\mathbb{H}$, $z \mapsto (z+1)/(z-1)$. Then composite it with the map $z \mapsto z^3$.

Ch8-Q5: If $w \in \mathbb{H}$, $z^2 + 2zw + 1 = 0$ has two distinct roots. Thus it is injective. Write $z = re^{it}$, where $0 \le r < 1$, $t \in (-\pi, \pi)$ then

$$\begin{aligned} -\frac{1}{2} Im(z + \frac{1}{z}) &= -\frac{1}{2} Im(re^{it} + \frac{1}{r}e^{-it}) \\ &= -\frac{1}{2}\sin t \cdot (\frac{r^2-1}{r}). \end{aligned}$$

This takes all value in $\mathbb{R}^+$.

Ch8-Q10: Consider the biholomorphism $\varphi : \mathbb{D} \to \mathbb{H}$ by $\varphi(z) = i\frac{1-z}{1+z}$. Then

$$G = F \circ \varphi : \mathbb{D} \to \overline{\mathbb{D}}$$

and $G(0) = 0$. Apply Schwarz Lemma to conclude that

$$|F(\varphi(z))| \le |z|, \quad \forall z \in \mathbb{D}.$$

By taking inverse of φ, we can conclude the desired result.

Ch8-Q12: (a) Suppose a, b are two distinct fixed point of f, consider the map $\phi : z \mapsto (z-a)/(1-\bar{a}z)$, the map $g = \phi \circ f \circ \phi^{-1}$ satisfies

$$g(0) = 0, \quad \text{and} \quad g : \mathbb{D} \to \mathbb{D}.$$

By Schwarz lemma,

$$|g(z)| \le |z| \quad \text{on } \mathbb{D}.$$

Since b is another fixed point of f and thus of g, so it implies that $g(z) = cz$ for some constant $|c| = 1$. More precisely,

$$g(\phi(b)) = \phi(b).$$

So, $c = 1$, and $f(z) = z$.

(b) No, consider the conformal map from $\mathbb{D}$ to $\mathbb{H}$. We can identify the unit disc with the upper half plane. But the translation $z \mapsto z - 1$ has no fixed point.

Ch8-Q13: (a) For w fixed. Denote $g(z) = \dfrac{f(z) - f(w)}{1 - \overline{f(w)}f(z)}$ and let $\varphi(z) = \dfrac{z+w}{1+\bar{w}z}$. Then $h = g \circ \varphi$ satisfies $h(0) = 0$ and $|h(z)| \le 1$ for all $|z| \le 1$. By Schwarz lemma,

$$|h(z)| \le |z|, \quad \forall\, |z| \le 1.$$

Therefore,

$$|g(z)| = |h(\varphi^{-1}(z))| \le |\varphi^{-1}(z)| = \left| \frac{z-w}{1-\bar{w}z} \right|.$$

By replacing f by f^{-1}, we have the full conclusion.

(b) Rearranging the inequality,

$$\left| \frac{f(z) - f(w)}{z - w} \right| \le \left| \frac{1 - \overline{f(w)}f(z)}{1 - \bar{w}z} \right|$$

Since f is complex differentiable, we may let $w \to z$ which implies

$$\frac{|f'(z)|}{1 - |f(z)|^2} \le \frac{1}{1 - |z|^2}.$$

Add ex Q1: $f_1 = \frac{z+1}{1-z}$ first map the upper half disk onto the first quadrant. Then $f_2(z) = z^2$ maps it to upper half plane. Using the conformal equivalence between the upper half plane and disk again.

Add ex Q2: By mean of conformal map, we can identify the half strip with $\{z \in \mathbb{C} : Re(z) > 0, Im(z) \in (0, \pi)\}$. Then try the map $z \mapsto \cosh z$.

Add ex Q3: We first show that the area of Ω is given by $\int_{\mathbb{D}} |f'(z)|^2 dxdy$.

$$|\Omega| = \int_{\Omega} dA$$

As f provides a parametrization for Ω, Write $f = u + iv$, we have

$$\int_{\Omega} dA = \int_{\mathbb{D}} \left| \frac{\partial(u,v)}{\partial(x,y)} \right| dxdy.$$

Let's compute the Jacobian matrix.

$$\begin{aligned} \left| \frac{\partial(u,v)}{\partial(x,y)} \right| &= |\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}| \\ &= |\frac{\partial u}{\partial x}\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial u}{\partial y}| \\ &= |\nabla u|^2 = |f'(z)|^2. \end{aligned}$$

Putting it back to obtain the result, we get

$$|\Omega| = \int_{\mathbb{D}} |f'(z)|^2 dxdy = \int_0^1 \int_0^{2\pi} |f'(re^{it})|^2 r dr dt.$$

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

be its power expansion at 0. Then

$$f'(re^{it}) = \sum_{n=1}^{\infty} a_n n r^{n-1} e^{it(n-1)} = \sum_{n=0}^{\infty} a_{n+1}(n+1) r^n e^{int}.$$

$$\begin{aligned} \int_0^1 \int_0^{2\pi} |f'(re^{it})|^2 r dr dt &= \int_0^1 \int_0^{2\pi} \sum_{m,n=0}^{\infty} a_{n+1}\overline{a}_{m+1}(n+1)(m+1) r^{m+n} e^{it(n-m)} r dr dt \\ &= \sum_{m,n=0}^{\infty} a_{n+1}\overline{a}_{m+1}(n+1)(m+1) \left(\int_0^1 r^{m+n+1} dr \right) \cdot \left(\int_0^{2\pi} e^{it(n-m)} dt \right) \\ &= \sum_{n=0}^{\infty} |a_{n+1}|^2 (n+1)^2 (2n+2)^{-1} \cdot 2\pi \\ &= \sum_{n=1}^{\infty} |a_n|^2 n\pi. \end{aligned}$$

Add ex Q4: Without loss of generality, we assume $a = 0, b = 1$. Let $M > C$ be a constant such that

$$|f(z)| \leq M \quad \text{on } \{z : Re(z) \in (0,1)\} = S.$$

And $|f(z)| \leq C$ on ∂S. We claim that $|f(z)| \leq C$ on S. Consider $g = f(z)e^{\epsilon z^2}$. On ∂S, when $z = iy$

$$|g(z)| \leq Ce^{-\epsilon y^2} \leq C$$

When $z = 1 + iy$,

$$|g(z)| \leq C|e^{\epsilon(1+iy)^2}| \leq Ce^{\epsilon(1-y^2)}.$$

When $z \in int(S)$, write $z = x + iy$ where $x \in (0,1), y \in (-\infty, +\infty)$,

$$|g(z)| \leq Me^{\epsilon(x^2-y^2)} \leq Me^{\epsilon(1-y^2)}$$

There exists $T > 0$ such that for all $|y| \geq T$, $Me^{\epsilon(1-y^2)} \leq C$.
Apply maximum principle on $\{z : Re(z) \in (0,1), |Im(z)| \leq T\} = \Omega$. We know that on Ω,

$$|g(z)| \leq \sup_{\partial\Omega} |g(w)| = Ce^{\epsilon}.$$

On the other hand, when $|Im(z)| > T$,

$$|g(z)| \leq C$$

Combine all results, we have

$$|g(z)| \leq Ce^{\epsilon} \quad \text{on } S$$

Which implies

$$|f(z)| \leq Ce^{\epsilon}e^{-\epsilon z^2}.$$

Letting $\epsilon \to 0$ to conclude it.

Add ex Q5: The spirit of the proof is essentially same with the content of tutorial 29/11 except that you are required to map everything back to strip in order to apply Three strip theorem. I here give a detailed proof in the tutorial.

Claim: $\lim_{|z|\to 1} |f(z)| = 1$ **or** r_2.

Suppose we can find $\{x_n\}, \{y_n\}$ so that $|y_n|, |x_n| \to 1$ but $|f(x_n)| < \sqrt{r_1}$ and $|f(y_n)| > \sqrt{r_1}$, for all n. By continuity, there exists $|z_n| \to 1$ so that $|f(z_n)| = \sqrt{r_1}$. But since $\overline{\partial B_{\sqrt{r_1}}}$ is compactly contained in $A(1, r_2)$. The inverse is still compact which is away from $\partial A(1, r_1)$. This contradicts with the existence of z_n. So we may assume that if $|z|$ close enough to 1, then $|f(z)|$ will be close to 1.

By Bolzano Weierstrass theorem and open mapping theorem, every $|z_n| \to 1$ has a convergent subsequence so that $|f(z_{n_k})| \to 1$. Therefore, the claim is true. Similarly, $\lim_{|z|\to r_1} |f(z)| = r_2$.

By three circle theorem or maximum principle on $\log |f|$, for any $z \in A(1, r_1)$,

$$\log |f| = \frac{\log |z|}{\log r_1} \log r_2.$$

That is $|f| = |z|^\alpha$, where $\alpha = \log r_2 / \log r_1$. Noted that z^α is not necessarily well defined on $A(1, r_1)$. If α is integer, then we can apply open mapping theorem and injectivity to conclude that $\alpha = 1$. To show this,

$$\begin{aligned}
\frac{f'(z)}{f(z)} &= \frac{\partial}{\partial z} \log f \\
&= \frac{\partial}{\partial z} \log |f|^2 \\
&= \alpha \frac{\partial}{\partial z} \log |z|^2 \\
&= \alpha z^{-1}.
\end{aligned}$$

Integrating over a sphere inside $A(1, r_1)$ and uses argument principle, we conclude that $\alpha \in \mathbb{Z}$.